15.4) 식 (15.11)을 증명하라.

$$E[\tilde{\sigma}^2] = \sigma^2 + \beta_r' X_r' [I - X_p (X_p' X_p)^{-1} X_p^T] X_r \beta_r / (n-p-1)$$

$X = [X_p, X_r]$

$y = X_p \beta_p + X_r \beta_r + \epsilon$

X_p : 절편과 p개의 독립변수를 포함

X_r : k-p 개의 독립 변수를 포함

$$\tilde{\sigma}^2 = y' [I - X_p (X_p' X_p)^{-1} X_p'] y / (n-p-1)$$

Reduced Model : $y = X_p \beta_p + \epsilon$

Full Model : $y = X_p \beta_p + X_r \beta_r + \epsilon$

$\tilde{\sigma}^2 = MSE(R)$

$E[\tilde{\sigma}^2] = \frac{1}{n-p-1} E[y'Ay]$, $A = I - X_p (X_p' X_p)^{-1} X_p'$

$E[y'Ay] = \text{trace}(A\Sigma) + \mu' A \mu$, $\Sigma = Var(y)$

$= \sigma^2 \text{trace}[I - X_p (X_p' X_p)^{-1} X_p') + (X\beta)' [I - X_p (X_p' X_p)^{-1} X_p'] (X\beta)$

$= \sigma^2 (n-p-1) + (X_p \beta_p + \Pi_{X_p} X_r \beta_r + \Pi_{X_{r\perp}} X_r \beta_r)' [I - X_p (X_p' X_p)^{-1} X_p'] (X_p \beta_p + \Pi_{X_p} X_r \beta_r + \Pi_{X_{r\perp}} X_r \beta_r)$

$= \sigma^2 (n-p-1) + (\Pi_{X_{r\perp}} X_r \beta_r)' (I - \Pi_{X_p}) (\Pi_{X_{r\perp}} X_r \beta_r)$

$= \sigma^2 (n-p-1) + \{(X_r - \Pi_{X_p} X_r) \beta_r\}^T (I - \Pi_{X_p}) \{(X_r - \Pi_{X_p} X_r) \beta\}$

$= \sigma^2 (n-p-1) + \beta_r' X_r' (I - \Pi_{X_p}) X_r \beta_r$

$\therefore E[\tilde{\sigma}^2] = \sigma^2 + \frac{\beta_r' X_r' (I - \Pi_{X_p}) X_r \beta_r}{(n-p-1)}$ □

Q15.5) 정리 (15.1)을 증명하라.

《정리 15.1》 행렬 $Var(b_p) - Var(\tilde{\beta}_p)$은 양반정치행렬 (positive semi-definite matrix) 이다.

$\hat{y} = X_p b_p + X_r b_r = \Pi_{X_p} y + \Pi_{X_{r\perp}} y$

$= X_p b_p + \Pi_{X_p} X_r b_r + \Pi_{X_{r\perp}} X_r b_r$

$\therefore \Pi_{X_p} y - \Pi_{X_p} X_r b_r = X_p b_p$

$b_p = (X_p' X_p)^{-1} X_p^{-1} y + (X_p' X_p)^{-1} X_p^T X_r b_r$

$= \tilde{\beta}_p + A b_r$, $A = (X_p' X_p)^{-1} X_p^T X_r$

$\tilde{\beta}_p = (X_p' X_p)^{-1} X_p^{-1} y$

$Var(b_p) = Var(\tilde{\beta}_p + A b_r)$

$= Var(\tilde{\beta}_p) + A\, Var(b_r) A^T + 2\, Cov(\tilde{\beta}_p, A b_r)$

$Cov(\tilde{\beta}_p, A b_r) = 0$ $\because \tilde{\beta}_r \in C(X_p)$, $b_r \in C(X_p)^\perp$

$\therefore Var(b_p) = Var(\tilde{\beta}_p) + A\, Var(b_r) A^T$

$Var(b_p) - Var(\tilde{\beta}_p) = A\, Var(b_r) A^T = A P \Lambda P^T A^T$

let $x \in \mathbb{R}^{p+1}$, $x^T \{Var(b_p) - Var(\tilde{\beta}_p)\} x = x^T A P \Lambda P^T A^T x$

$= \| \Lambda^{1/2} P^T A^T x \|^2 \geq 0$

$\therefore Var(b_p) - Var(\tilde{\beta}_p)$ is positive semi-definite.

15.6) 정리 (15.2)를 증명하라.

≪정리 15.2≫ $Var(\hat{y}) \geq Var(\tilde{y}_p)$

$\hat{y} = X(X^TX)^{-1}X^Ty$

$\tilde{y}_p = X_p(X_p^TX_p)^{-1}X_p'y$

$Var(\hat{y}) = \Pi_X Var(y) \Pi_X^T = \sigma^2\Pi_X = \sigma^2 X(X'X)^{-1}X'$

$Var(\tilde{y}_p) = \Pi_{X_p} Var(y) \Pi_{X_p}^T = \sigma^2 \Pi_{X_p}$

$Var(\hat{y}) - Var(\tilde{y}_p) = \sigma^2(\Pi_X - \Pi_{X_p}) = \sigma^2 \Pi_{X_{r,\perp}}$

$* \Pi_{X_{r,\perp}} = X_{r,\perp}(X_{r,\perp}^TX_{r,\perp})X_{r,\perp}^T$

$X_{r,\perp} = X_r - \Pi_{X_p}X_r$

Suppose $c \in \mathbb{R}^n$ is an arbitrary vector

$c^T(\sigma^2\Pi_{X_{r,\perp}})c = \sigma^2(c^T\Pi_{X_{r,\perp}}^T)(\Pi_{X_{r,\perp}}c)$

$= \|\sigma \Pi_{X_{r,\perp}} c\|^2 \geq 0$

$\therefore$ $Var(\hat{y}) - Var(\tilde{y}_p)$ is a positive semi-definite matrix.

8(15.7) 정리 (15.3)을 증명하라.

≪정리 15.3≫ 만약 행렬 $Var(b_r) - \beta_r\beta_r'$이 양반정치행렬이면

$Var(b_p) - MSE(\tilde{\beta}_p)$도 양반 정치행렬이며,

$Var(\tilde{y}) \geq MSE(\tilde{y}_p)$이 성립한다.

$\hat{y} = X_pb_p + X_rb_r = \Pi_{X_p}y + \Pi_{X_{r,\perp}}y$

$= X_p(X_p^TX_p)^{-1}X_p^Ty + X_{r,\perp}(X_{r,\perp}^TX_{r,\perp})^{-1}X_{r,\perp}^Ty$

$\hookrightarrow X_pb_p + \Pi_{X_p}X_rb_r + \Pi_{X_{r,\perp}}X_rb_r$

$X_pb_p + \Pi_{X_p}X_rb_r = \Pi_{X_p}y$

$\therefore b_p = (X_p'X_p)^{-1}X_p'y - (X_p'X_p)^{-1}X_p^TX_rb_r$

$Var(b_p) = Var((X_p'X_p)^{-1}X_p'y - (X_p'X_p)^{-1}X_p^TX_rb_r)$, $b_r = (X_{r,\perp}'X_{r,\perp})^{-1}X_{r,\perp}'y$

$MSE(\tilde{\beta}_p) = E[(\tilde{\beta}_p - \beta_p)^2] = E[(\tilde{\beta}_p - E[\tilde{\beta}_p] + E[\tilde{\beta}_p] - \beta_p)^2]$

$= E[(\tilde{\beta}_p - E[\tilde{\beta}_p])^2] + E[bias(\tilde{\beta}_p)^2] + 2E[(\tilde{\beta}_p - E[\tilde{\beta}_p])(E[\tilde{\beta}_p] - \beta_p)^T]$

$= Var(\tilde{\beta}_p) + bias(\tilde{\beta}_p)bias(\tilde{\beta}_p)^T$

$bias(\tilde{\beta}_p) = E[\tilde{\beta}_p] - \beta_p = E[b_p + Ab_r] - \beta_p = Ab_r$, $A = (X_p'X_p)^{-1}X_p'X_r$

$\therefore bias(\tilde{\beta}_p)bias(\tilde{\beta}_p)^T = Ab_rb_r'A'$

$Var(b_p) = Var(\tilde{\beta}_p - Ab_r) = Var(\tilde{\beta}_p) + AVar(b_r)A' - 2Cov(\tilde{\beta}_p, b_r)$

$= Var(\tilde{\beta}_p) + AVar(b_r)A'$ ($\because \tilde{\beta}_p \in C(X_p)$, $b_r \in C(X_p)^\perp$)

$Var(b_p) - MSE(\tilde{\beta}_p) = Var(\tilde{\beta}_p) + AVar(b_r)A' - Var(\tilde{\beta}_p) - A\beta_r\beta_r'A'$

$= A(Var(b_r) - \beta_r\beta_r')A' \geq 0$

$\because$ $Var(b_r) - \beta_r\beta_r'$ is semi-positive definite.

15.8. 식 (15.36)을 증명하라.

$Q = \int_R [Var(\hat{y}) - MSE(\tilde{y}_p)]\, dW(r) = \frac{1}{n}\sum_{i=1}^{n}[Var(\hat{y}_j) - MSE(\tilde{y}_{p_j})]$ … (15.35)

만약 $Var(b_r) = B\sigma^2$라 놓으면, 식 (15.35)는 다음과 같이 된다.

$\frac{1}{n}\sum_{j=1}^{n}[Var(\hat{y}_j) - MSE(\tilde{y}_{p_j})] = \frac{r}{n}\sigma^2\left(1 - \frac{\beta_r'B^{-1}\beta_r}{r\sigma^2}\right)$ … (15.36)

* y_j : 실제의 측정값

* $\tilde{y}_{p_j}$: 회귀방정식에 의한 추정값. $\tilde{y}_p = X_p\tilde{\beta}_p$

$\frac{1}{n}\sum_{j=1}^{n}[Var(\hat{y}_j) - MSE(\tilde{y}_{p_j})]$

$\frac{1}{n}\{trace(Var(y)) - trace(MSE(\hat{y}_p))\}$

$= \frac{1}{n}\{trace(\sigma^2I_n) - trace(Var(\hat{y}_p) + Bias(\hat{y}_p)Bias(\hat{y}_p)^T)\}$

$* \hat{y}_p = X_p(X_p^TX_p)^{-1}X_p^Ty$

$Var(\hat{y}_p) = \sigma^2(X_p^TX_p)^{-1}$

$trace(\hat{y}_p) = \sigma^2 trace((X_p^TX_p)^{-1})$

$= \sigma^2 rank((X_p'X_p)) = \sigma^2p$

$Bias(\hat{y}_p) = E[y] - E[\hat{y}_p] = X_p\beta_p + X_r\beta_r - E[\Pi_{X_p}y]$

$= X_p\beta_p + X_r\beta_r - \Pi_{X_p}(X_p\beta_p + X_r\beta_r)$

$= (I_n - \Pi_{X_p})X_r\beta_r$

$Bias(\hat{y}_p)Bias(\hat{y}_p)^T = [(I_n - \Pi_{X_p})X_r\beta_r][(I_n - \Pi_{X_p})X_r\beta_r]^T$

$trace(Bias(\hat{y}_p)Bias(\hat{y}_p)^T) = trace(\beta_r'X_r'(I_n - \Pi_{X_p})X_r\beta_r)$

$= \beta_r'X_r'(I_n - \Pi_{X_p})X_r\beta_r$

$= \beta_r'X_r'\Pi_{X_{r,\perp}}X_r\beta_r$

$= \beta_r'X_{r,\perp}'\Pi_{X_{r,\perp}}X_{r,\perp}\beta_r$

$= \beta_r'X_{r,\perp}'X_{r,\perp}(X_{r,\perp}'X_{r,\perp})^{-1}X_{r,\perp}'X_{r,\perp}\beta_r$

$= \beta_r'(X_{r,\perp}'X_{r,\perp})\beta_r$

$Var(b_r) = Var((X_{r,\perp}^TX_{r,\perp})^{-1}X_{r,\perp}^Ty)$

$= \sigma^2(X_{r,\perp}^TX_{r,\perp})^{-1}$

$B\sigma^2 = Var(b_r)$ $\therefore B = (X_{r,\perp}^TX_{r,\perp})^{-1}$ $\therefore B^{-1} = X_{r,\perp}'X_{r,\perp}$

$\therefore \beta_r'B^{-1}\beta_r = \beta_r'(X_{r,\perp}'X_{r,\perp})\beta_r$

$= trace(Bias(\hat{y}_p)Bias(\hat{y}_p)^T)$

$\frac{1}{n}\{trace(\sigma^2I_n) - trace(Var(\hat{y}_p) + Bias(\hat{y}_p)Bias(\hat{y}_p)')$

$= \frac{1}{n}\{n\sigma^2 - p\sigma^2 - \beta_r'B^{-1}\beta_r\}$

$= \frac{\sigma^2 r}{n}\left\{1 - \frac{\beta_r'B^{-1}\beta_r}{r\sigma^2}\right\}$ ✓✓

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15.9.) 다음을 유도하여라.

$$Q = \int_R [\,\mathrm{Var}(\hat{y}) - \mathrm{MSE}(\tilde{y}_p)\,]\,dW(r)$$

$$= \sigma^2\{\,\mathrm{tr}[(x'x)^{-1}M] - \mathrm{tr}[(x_p'x_p)^{-1}M_{pp}]\} - \beta_r'[\,A'M_{pp}A - 2A'M_{pr} + M_{rr}]\beta_r$$

* $M = \int_R xx'\,dW(r)$, $M_{ij} = \int_R x_i x_j'\,dW(r)$

$$Q = \int_R [\,\mathrm{Var}(\hat{y}) - \mathrm{MSE}(\tilde{y}_p)\,]\,dW(r)$$

$$= \int_R [\,\sigma^2 x'(X^TX)^{-1}x - x_p'(x_p'x_p)^{-1}x_p\cdot\sigma^2 - \beta_r'[(x_p'A - x_r')(x_p'A - x_r')]\beta$$

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★ 15.10) 뒤로부터 제거하는 방법 (backward elimination method) 에서

부분 F-검정치

$$F_o = \frac{b_i^2}{\mathrm{Var}(b_i)}$$

이 15.2.2 에 설명되어 있는 부분 F 검정치

$$F_o = \frac{\mathrm{SSE}(R) - \mathrm{SSE}(F)}{\mathrm{MSE}}$$

와 같음을 증명하라.

$$\mathrm{SSE}(R) - \mathrm{SSE}(F) = (A\hat{\beta})^T[A(X^TX)^{-1}A^T]^{-1}(A\hat{\beta})$$

$A = e_i^T$. Let $A(X^TX)^{-1}A^T = e_i^T(X^TX)^{-1}e_i = c_{ii}$

$$\mathrm{Var}(b_i) = \mathrm{Var}(e_i^Tb) = e_i^T\mathrm{Var}(b)e_i = \sigma^2 e_i^T(X^TX)^{-1}e_i$$
$$= \sigma^2 c_{ii}$$

$$\mathrm{MSE} = \mathrm{SST}/(n-p-1) = y^T(I_n - \Pi_X)y/(n-p-1)$$
$$= \hat{\sigma}^2$$

$$\therefore \widehat{\mathrm{Var}(b_i)} = \mathrm{MSE}\cdot c_{ii} \Leftrightarrow \mathrm{MSE} = \widehat{\mathrm{Var}(b_i)}\cdot c_{ii}^{-1}$$

$$\therefore \frac{\mathrm{SSE}(R) - \mathrm{SSE}(F)}{\mathrm{MSE}} = \frac{(A\hat{\beta})^T[A(X^TX)^{-1}A^T]^{-1}(A\hat{\beta})}{\widehat{\mathrm{Var}(b_i)}\cdot c_{ii}^{-1}}$$

$$= \frac{b_i\cdot c_{ii}^{-1}\cdot b_i}{\widehat{\mathrm{Var}(b_i)}\cdot c_{ii}^{-1}} = \frac{b_i^2}{\widehat{\mathrm{Var}(b_i)}}$$